THE VOLUME OF A SPHERE*

Granted—If a moving plane be constantly perpendicular to a fixed line which passes through a given point of the plane, any two figures in the plane which are equal in area will generate equal volumes.

Corollary—If the two figures in the plane are not equal, the two volumes generated will be to each other as their generating areas in the plane.

Theorem—The volume of a sphere is equal to one-sixth of a cube upon its diameter multiplied by the circummetric ratio · π.

In the figure, let ABCD — E be a sphere, AC its diameter, V its volume, and π the circummetric ratio.

$$\text{Then will} \qquad V = \frac{\pi}{6}\overline{AC}^3$$

Let ACHG be a rectangle whose plane cuts the sphere in the great circle ABCD. In this plane produce $\overline{CH}$ to K, taking HK = HG, and draw GK; also, through G, perpendicular to the plane GHK, draw GL = GH, and draw LH and LK. Then will GHK and HGL be two right triangles, right angled at H and G, respectively and, having their planes perpendicular to each other.

* Some years ago Prof. J. B. Johnson turned over to me a trunk fairly well filled with papers and manuscripts originally belonging to and largely written by Alonzo Jackman, Adjutant General of Vermont during the Civil War and a professor at Norwich University from 1839 till he died, during the Winter of 1879. (During the last five of those years Johnson and I were in his classes in mathematics.) Among the manuscripts were: a trigonometry, a treatise on conic sections, a detailed description of an astronomical telescope with revolving mercurial reflective very much like the one invented nearly thirty years later; and this demonstration. I had not the time then to examine all the papers with care; but I had copies made of some, including the "Volume of a Sphere," which follows essentially as written, save the omission of certain references to texts long ago discarded if not forgotten.

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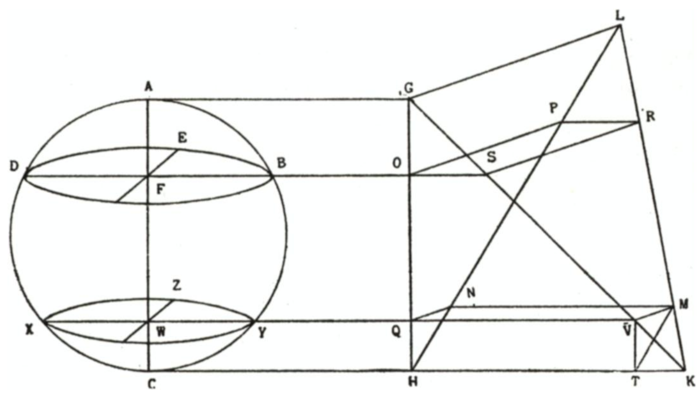
The planes GHK, GHL, KHL and KGL form a pryamid whose base is GHK, whose altitude is $\overline{GL}$, and whose volume $\frac{1}{2}\overline{GH}\cdot\overline{HK}\cdot\frac{1}{3}\overline{GL}$. Or, since $\overline{GH}=\overline{HK}=\overline{GL}=\overline{AC}$ by supposition, the pyramidal volume, GHK — L $=\frac{1}{6}\overline{AC}^3$.



Through any point of $\overline{AC}$, as F, draw a plane perpendicular to AC, or GH, and denote the sections it cuts in the sphere and the pyramid by DEB and OPRS. GL and HK will be parallel to this plane, and DEB will be a circle whose center is F.

Since $\overline{OP}$ and $\overline{SR}$ are both parallel to $\overline{GL}$ they are parallel to each other; and for the same reason $\overline{OS}$ and $\overline{PR}$, parallel to $\overline{HK}$, are parallel to each other. Therefore, OPRS is a rectangle, right-angled at O. Further, since $\overline{GH}=\overline{HK}=\overline{GL}$, we have in the triangles GHK and HGL, $\overline{GO}=\overline{OS}$ and $\overline{HO}=\overline{OP}$, whence $\overline{GO}\cdot\overline{OH}=\overline{OS}\cdot\overline{OP}=$ area OPRS. But $\overline{GO}\cdot\overline{OH}=\overline{AF}\cdot\overline{FC}=\overline{FD}^2$. Consequently the area OPRS $=\overline{FD}^2$.

Now $\pi\,\overline{FD}^2$ equals the area of the circular section DEB, and, therefore, the section DEB $=\pi\cdot$ OPRS.

Since the same ratio will exist between any two corresponding sections of the sphere and the pyramid wherever the plane

cuts AC, if we conceive the plane to move continuously through all points from C to A carrying with it these corresponding sections, one of them will generate the sphere and the other the pyramid, and the sphere will be π times the volume of the pyramid; or

$$V = \frac{\pi}{6} \overline{AC}^3$$

COROLLARY

Any segment of the sphere cut off by a plane perpendicular to AC will have a volume equal to π times the corresponding portion of the pyramid—or, in the figure, the segment $XYZ - C = \pi \cdot QNMV - KH$.

Pass a plane through MV parallel to QNH cutting HK at T.

The volume of the pyramid $MVT - K = \frac{\overline{VM} \cdot \overline{VT}}{2} \cdot \frac{\overline{TK}}{3}$

$$= \frac{\overline{VM} \cdot \overline{VT} \cdot \overline{TK}}{6}.$$

The volume of the prism $QNH - V = \frac{\overline{MV} \cdot \overline{VT} \cdot VQ}{2}$.

And the volume

$$QNMV - KH = \overline{VM} \cdot \overline{VT} \cdot \left(\frac{VQ}{2} + \frac{TK}{6}\right)$$

Now since $\overline{HQ} = \overline{QN} = \overline{VT} = \overline{CW}$, and $\overline{VQ} = \overline{QG} = \overline{AW}$, if we denote $\overline{AC}$ by 2r and CW by h, the equation above becomes

Volume

$$XYZ - C = \pi ONMV - KH =$$

$$\pi\overline{VM} \cdot \overline{VT} \left(\frac{\overline{VQ}}{2} + \frac{\overline{TK}}{6}\right) = \pi h^2 \left(r - \frac{h}{3}\right)$$

Or:

The volume of a spherical segment equals the volume of a cylinder whose radius is the height of the segment and whose altitude is the radius of the sphere diminished by one-third the height of the segment.